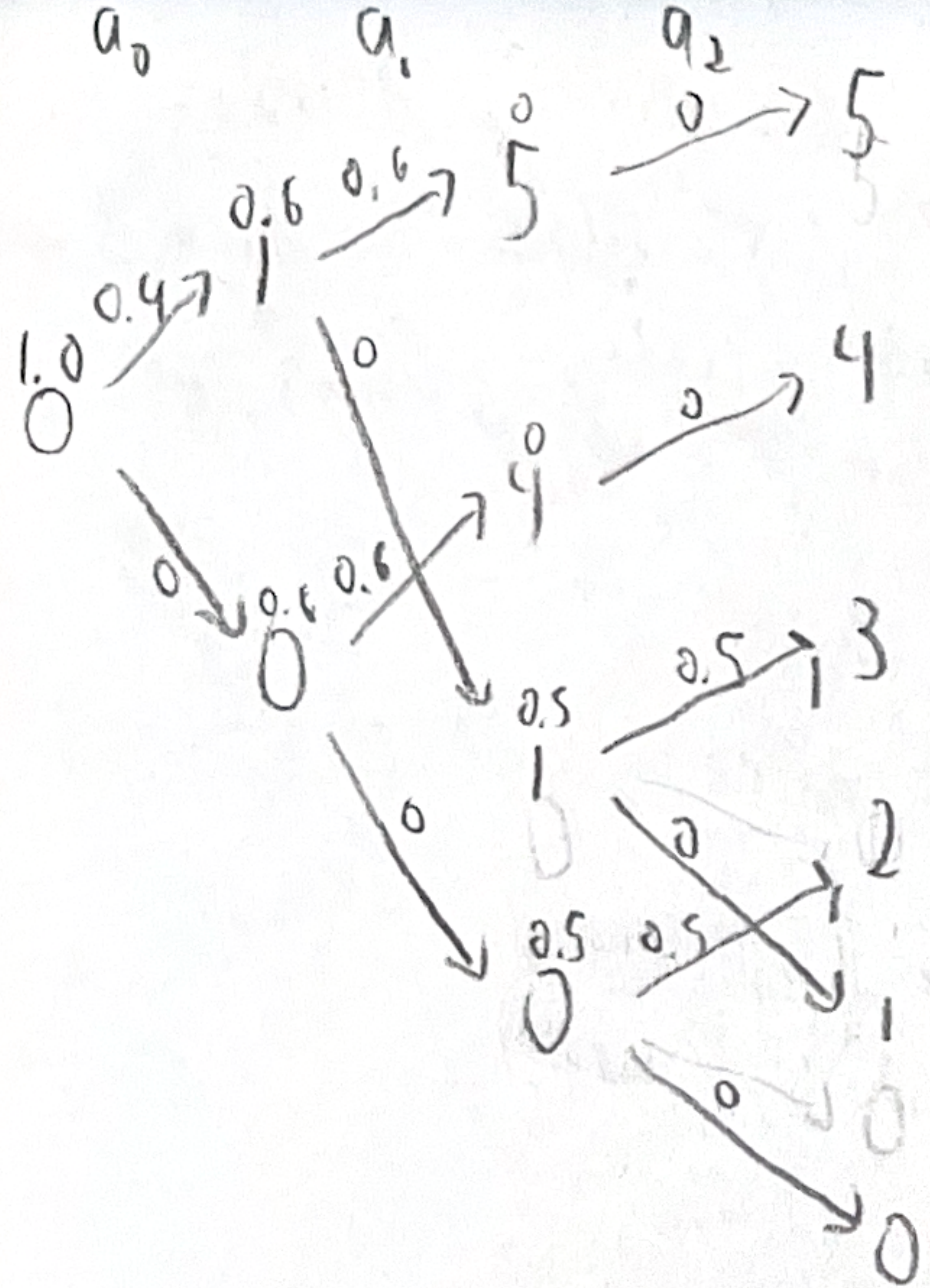

Take paths that maximize reward + reward to go

a_0 gives $1 \to 0.4 + 0.6 = 1$

$0 \to 0 + 0.6 = 0.6$ so $a_0 = 1$

a_1 gives $1 \to 0.6 + 0 = 0.6$

$0 \to 0 + 0.5 = 0.5$ so $a_1 = 1$

optimal value is 1

a_2 only has the choice 0 so $a_2 = 0$

1) $V_3^*(x_3) = \begin{cases} 1, & x_3 \geq 5 \\ 0, & x_3 = 1,2,3,4 \end{cases}$

$V_k^*(x_k) = \max_u E\left[0.4 \cdot V_{k+1}^*(x_k+u) + 0.2\, V_{k+1}^*(x_k) + 0.4\, V_{k+1}^*(x_k-u)\right]$

2) $V_2^*(0) = 0$ $\quad V_2^*(1) = 0$ $\quad V_2^*(2) = 0$ $\quad \overset{\text{bet 2 or 3}}{V_2^*(3) = 0.4}$ $\quad \overset{\text{bet 1-4}}{V_2^*(4) = 0.4}$ $\quad \overset{\text{bet 0}}{V_2^*(5+) = 1}$

$V_1^*(0) = 0$ $\quad V_1^*(1) = 0$ $\quad \overset{\text{bet 1 or 2}}{V_1^*(2) = 0.16}$ $\quad V_1^*(5+) = 1$

$V_1^*(3) = \max \begin{cases} 0 \to 0.4 \\ 1 \to 0.4 \cdot 0.4 + 0.2 \cdot 0.4 + 0.4 \cdot 0 = 0.24 \\ 2 \to 0.4 \cdot 1 + 0.2 \cdot 0.4 + 0.4 \cdot 0 = 0.48 \\ 3 \to 0.4 \cdot 1 + 0.2 \cdot 0.4 + 0.4 \cdot 0 = 0.48 \end{cases}$

$V_1^*(4) = \max \begin{cases} 0 \to 0.4 \\ 1 \to 0.4 \cdot 1 + 0.2 \cdot 0.4 + 0.4 \cdot 0.4 = 0.64 \\ 2 \to 0.4 \cdot 1 + 0.2 \cdot 0.4 + 0.4 \cdot 0 = 0.48 \\ 3 \to 0.4 \cdot 1 + 0.2 \cdot 0.4 + 0.4 \cdot 0 = 0.48 \\ 4 \to 0.4 \cdot 1 + 0.2 \cdot 0.4 + 0.4 \cdot 0 = 0.48 \end{cases}$

$V_0^*(3) = \max \begin{cases} 0 \to 0.48 \\ 1 \to 0.4 \cdot 0.64 + 0.2 \cdot 0.48 + 0.4 \cdot 0.16 = 0.416 \\ 2 \to 0.4 \cdot 1 + 0.2 \cdot 0.48 + 0.4 \cdot 0 = 0.496 \\ 3 \to 0.4 \cdot 1 + 0.2 \cdot 0.48 + 0.4 \cdot 0 = 0.496 \end{cases}$

current money \ Turn	0	1	2
0	NA	~	~
1	NA	~	~
2	NA	1/2	~
3	2/3	2/3	2/3
4	NA	1	1-4
5+	NA	0	0

~ ⇒ always lose

1/2 ⇒ bet one or two. Doesn't matter.

1-4 ⇒ bet 1, 2, 3, 4. Doesn't matter.

(1) $V_k^*(x_k) = \max \begin{cases} x_k - p \\ E[V_{k+1}^*(x_k + w_k)] \end{cases}$ $\qquad V_N^*(x) = \begin{cases} x - p & \text{if } x > p \\ 0 & \text{if } x \le p \end{cases} = \max \begin{cases} x-p \\ 0 \end{cases}$

(2) $V_{N-1}^*(x) = \max \begin{cases} x - p \\ E[V_N^*(x + w_k)] \end{cases}$ $\qquad x - p = x - p$

$E[V_N^*(x - w_k)] > 0$ since $V_N^*(x) \ge 0$ $\Rightarrow V_{N-1}^*(x) \ge V_N^*(x)$ since

$b > c \Rightarrow \max(a, b) \ge \max(a, c)$

Inductive step:

Assume $V_{k+1}^*(x) \ge V_{k+2}^*(x)$

$V_k^*(x) = \max \begin{cases} x - p \\ E[V_{k+1}^*(x + w_k)] \end{cases}$ $\qquad V_{k+1}^*(x) = \max \begin{cases} x - p \\ E[V_{k+2}^*(x + w_{k+1})] \end{cases}$

$x - p = x - p$ $\quad$ since w_k and w_{k+1} have the same distribution and $V_{k+1}^*(x+w) \ge V_{k+2}^*(x+w)$ by the hypothesis,

$E[V_{k+1}^*(x + w_k)] \ge E[V_{k+2}^*(x + w_{k+1})]$

since $b > c \Rightarrow \max(a, b) \ge \max(a, c)$ $\quad V_k^*(x) \ge V_{k+1}^*(x)$ as needed

(3) $V_k^*(x) - x = \begin{cases} -p \\ E[V_{k+1}^*(x + w_k)] - x \end{cases}$ $\qquad E[V_{k+1}^*(x + w_k)] - x = E[w_k] + E[V_{k+1}^*(x + w_k) - (x + w_k)]$

Base $V_N^*(x) - x = \max \begin{cases} -p \\ -x \end{cases}$ $\qquad$ Assume $x_\ell > x_s \Rightarrow -x_\ell \le -x_s \Rightarrow \max\begin{Bmatrix} -p \\ -x_\ell \end{Bmatrix} \le \max\begin{Bmatrix} -p \\ -x_s \end{Bmatrix} = V_N^*(x_\ell) - x_\ell \le V_N^*(x_s) - x_s$

$V_k^*(x_\ell) - x_\ell = \begin{cases} -p \\ E[w_k] + E[V_{k+1}^*(x_\ell + w_k) - (x_\ell + w_k)] \end{cases}$ $\qquad V_k^*(x_s) - x_s = \begin{cases} -p \\ E[w_k] + E[V_{k+1}^*(x_s + w_k) - (x_s + w_k)] \end{cases}$

By the hypothesis $E[V_{k+1}^*(x_\ell + w_k) - (x_\ell + w_k)] \le E[V_{k+1}^*(x_s + w_k) - (x_s + w_k)]$

$\Rightarrow V_k^*(x_\ell) - x_\ell \le V_k^*(x_s) - x_s$ as needed

$$
\begin{array}{cc|ccc}
 & & & \text{to} & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 1/3 & 1/3 & 1/3 \\
\text{From} & 2 & 1/2 & 0 & 1/2 \\
 & 3 & 1 & 0 & 0
\end{array}
$$

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

stationary $\Rightarrow \pi = \pi P$ or $\pi(I-P)=0$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} a &= \tfrac{1}{3}a + \tfrac{1}{2}b + c = \tfrac{1}{3} \\ b &= \tfrac{1}{3}a \\ c &= \tfrac{1}{3}a + \tfrac{1}{2}b = \tfrac{1}{3}a + \tfrac{1}{6}a = \tfrac{3}{6}a = \tfrac{1}{2}a \end{aligned}$$

$\Rightarrow \pi = [a \;\; 1/3\,a \;\; 1/2\,a]$ and $\sum_{i=1}^{3} \pi_i = 1 \Rightarrow a + \frac{1}{3}a + \frac{1}{2}a = \frac{11}{6}a \Rightarrow a = \frac{6}{11}$ Let $a=1 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T = \begin{bmatrix} a \\ \frac{1}{3}a \\ \frac{1}{3}a \end{bmatrix}^T = \begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix}^T$

$\Rightarrow \pi = \left[\frac{6}{11} \;\; \frac{6}{33} \;\; \frac{6}{22}\right]$